Serre-Swan theorem for non-commutative C*-algebras. Revised edition[1]

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Abstract

We generalize the Serre-Swan theorem to non-commutative C*-algebras. For a Hilbert C*-module X over a C*-algebra $\mathcal{A}$, we introduce a hermitian vector bundle $\mathcal{E}_X$ associated to X. We show that there is a linear subspace Γ_X of the space of all holomorphic sections of $\mathcal{E}_X$ and a flat connection D on $\mathcal{E}_X$ with the following properties: (i) Γ_X is a Hilbert $\mathcal{A}$-module with the action of $\mathcal{A}$ defined by D, (ii) the C*-inner product of Γ_X is induced by the hermitian metric of $\mathcal{E}_X$, (iii) $\mathcal{E}_X$ is isomorphic to an associated bundle of an infinite dimensional Hopf bundle, (iv) Γ_X is isomorphic to X.

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1 Introduction

The Serre-Swan theorem [9, 15, 16] is described as follows:

Theorem 1.1 *Let Ω be a connected compact Hausdorff space and let $C(\Omega)$ be the algebra of all complex-valued continuous functions on Ω. Assume that X is a module over $C(\Omega)$. Then X is finitely generated projective iff there is a complex vector bundle E over Ω such that X is isomorphic onto the module of all continuous sections of E.*

By Theorem 1.1, finitely generated projective modules over the commutative C*-algebra $C(\Omega)$ and complex vector bundles over Ω are in one-to-one correspondence up to isomorphism. In non-commutative geometry [6, 17], a certain module over a non-commutative C*-algebra $\mathcal{A}$ is treated as a non-commutative vector bundle over the non-commutative space $\mathcal{A}$, generalizing Theorem 1.1 in a sense of *point-less* geometry. Therefore both a non-commutative space and a non-commutative vector bundle are invisible even if one desires to look hard.

[1]Original paper [11]. The essential mathematical statement is same as before.
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On the other hand, for a unital generally non-commutative C*-algebra $\mathcal{A}$, the functional representation on a certain geometrical space is studied by [4]. We review it as follows.

Definition 1.2 *A triplet $(\mathcal{P}, p, B)$ is the uniform Kähler bundle associated with $\mathcal{A}$ if $\mathcal{P}$ (= $\mathrm{Pure}\mathcal{A}$) is the set of all pure states of $\mathcal{A}$, endowed with the w^*-uniformity, i.e. the uniformity which induces the w^*-topology, B (= $\mathrm{Spec}\mathcal{A}$) is the spectrum of $\mathcal{A}$, the set of all equivalence classes of irreducible representations of $\mathcal{A}$, and p is the natural projection from $\mathcal{P}$ onto B by the GNS representation.*

For each $b \in B$, the fiber $\mathcal{P}_b \equiv p^{-1}(b)$ is a Kähler manifold (Appendix D in [4]). Especially, if $\mathcal{A}$ is commutative, then $\mathcal{P} \cong B$ and it is a compact Hausdorff space. In this case, each fiber of $(\mathcal{P}, p, B)$ is a 0-dimensional Kähler manifold. Define $C^\infty(\mathcal{P})$ the set of all fiberwise-smooth complex-valued functions on $\mathcal{P}$. The product $*$ on $C^\infty(\mathcal{P})$ is defined by

$$l * m \equiv l \cdot m + \sqrt{-1} X_m l \quad (l, m \in C^\infty(\mathcal{P})) \tag{1.1}$$

where X_l is the holomorphic part of the complex Hamiltonian vector field of l with respect to the Kähler form on $\mathcal{P}$. Then $C^\infty(\mathcal{P})$ is a * algebra with the unit $\mathbf{1}$ and the involution * by complex conjugation, which is not associative in general. Define the subset $C_u^\infty(\mathcal{P})$ of $C^\infty(\mathcal{P})$ consisting of uniformly continuous functions on $\mathcal{P}$.

Theorem 1.3 *For a unital non-commutative C*-algebra $\mathcal{A}$, the Gel'fand representation*

$$f_A(\rho) \equiv \rho(A) \quad (A \in \mathcal{A},\, \rho \in \mathcal{P}), \tag{1.2}$$

gives an injective * homomorphism f from $\mathcal{A}$ into $C^\infty(\mathcal{P})$ where $C^\infty(\mathcal{P})$ is endowed with the $*$-product in (1.1). The norm $\|\cdot\|$ on $f(\mathcal{A})$ defined by

$$\|l\| \equiv \sup_{\rho \in \mathcal{P}} \left|\left(\bar{l} * l\right)(\rho)\right|^{\frac{1}{2}} \quad (l \in f(\mathcal{A})), \tag{1.3}$$

is a C-norm on the associative * subalgebra $f(\mathcal{A})$.*

Furthermore $f(\mathcal{A})$ is precisely the subset $\mathcal{K}_u(\mathcal{P})$ of $C_u^\infty(\mathcal{P})$ defined by

$$\mathcal{K}_u(\mathcal{P}) \equiv \{l \in C_u^\infty(\mathcal{P}) : \bar{l} * l, l * \bar{l} \in C_u^\infty(\mathcal{P}),\ D^2 l = \bar{D}^2 l = 0\} \tag{1.4}$$

where D, $\bar{D}$ are the holomorphic and anti-holomorphic part, respectively, of covariant derivative of Kähler metric defined on each fiber of $\mathcal{P}$. In consequence, the following equivalence of C-algebras holds:*

$$\mathcal{A} \cong \mathcal{K}_u(\mathcal{P}).$$

Proof. See Proposition 3.2 in [4]. ∎

By Theorem 1.3, it seems that there exists a *geometry consisting of points* associated with not only a commutative C*-algebra but also a non-commutative one. According to Theorem 1.3, we introduce a representation of a Hilbert C*-module as the sections of a vector bundle over $\mathcal{P}$.

A vector space X is a *Hilbert C*-module* [7, 13] over a C*-algebra $\mathcal{A}$ if X is a right $\mathcal{A}$-module with an $\mathcal{A}$-valued inner product $\langle\cdot|\cdot\rangle$ which satisfies $\langle\eta|\xi a\rangle = \langle\eta|\xi\rangle a$ for each $\eta, \xi \in X$ and $a \in \mathcal{A}$, and X is complete with respect to the norm $\|\cdot\|$ defined by $\|\xi\| \equiv \|\langle\xi|\xi\rangle\|^{1/2}$ for $\xi \in X$.

Definition 1.4 *The triplet $(\mathcal{E}_X, \Pi_X, \mathcal{P})$ is the atomic bundle associated with a Hilbert C*-module X over a unital C*-algebra $\mathcal{A}$ if it is the fiber bundle with the base space $\mathcal{P}$ and the total space $\mathcal{E}_X$:*

$$\mathcal{E}_X \equiv \bigcup_{\rho\in\mathcal{P}} \mathcal{E}_{X,\rho}$$

where Π_X is the natural projection from $\mathcal{E}_X$ onto $\mathcal{P}$, and the fiber $\mathcal{E}_{X,\rho}$ for $\rho \in \mathcal{P}$ is the Hilbert space defined as follows: Define the quotient vector space $\mathcal{E}^o_{X,\rho} \equiv X/N_\rho$ where N_ρ is the closed subspace of X defined by $N_\rho \equiv \{\, \xi \in X : \rho(\|\xi\|^2) = 0 \,\}$. Define the inner product $\langle\cdot|\cdot\rangle_\rho$ on $\mathcal{E}^o_{X,\rho}$ by

$$\langle[\xi]_\rho|[\eta]_\rho\rangle_\rho \equiv \rho(\langle\xi|\eta\rangle) \qquad (\, [\xi]_\rho, [\eta]_\rho \in \mathcal{E}^o_{X,\rho}\,) \tag{1.5}$$

where $[\xi]_\rho \equiv \xi + N_\rho \in \mathcal{E}^o_{X,\rho}$ for $\xi \in X$. Let $\mathcal{E}_{X,\rho}$ denote the completion of $\mathcal{E}^o_{X,\rho}$ by the norm $\|\cdot\|_\rho$ associated with $\langle\cdot|\cdot\rangle_\rho$.

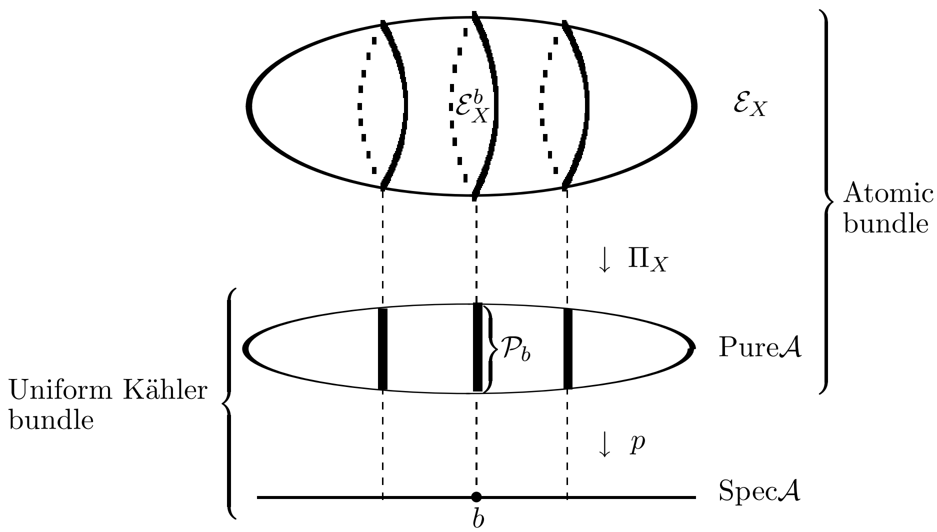
We show the property of $\mathcal{E}_X$. Let $\mathcal{H}$ denote a complex Hilbert space with $1 \leq \dim\mathcal{H} \leq \infty$. A triplet $(S(\mathcal{H}), \mu, \mathcal{P}(\mathcal{H}))$ is *the Hopf (fiber) bundle over $\mathcal{H}$* if *the projective Hilbert space $\mathcal{P}(\mathcal{H})$* and *the Hilbert sphere $S(\mathcal{H})$* are defined by

$$\mathcal{P}(\mathcal{H}) \equiv (\mathcal{H} \setminus \{0\})/\mathbf{C}^\times, \quad S(\mathcal{H}) \equiv \{z \in \mathcal{H} : \|z\| = 1\} \tag{1.6}$$

and the projection μ from $S(\mathcal{H})$ onto $\mathcal{P}(\mathcal{H})$ is defined by $\mu(z) \equiv [z]$ for $z \in S(\mathcal{H})$.

Theorem 1.5 *For $b \in B$ $(= \mathrm{Spec}\mathcal{A})$, let $\mathcal{H}_b$ be a representative of b, $\mathcal{E}^b_X \equiv \Pi^{-1}_X(\mathcal{P}_b)$ and $\Pi^b_X \equiv \Pi_X|_{\mathcal{E}^b_X}$. Then $(\mathcal{E}^b_X, \Pi^b_X, \mathcal{P}_b)$ is a locally trivial vector bundle which is isomorphic to the associated bundle of $(S(\mathcal{H}_b), \mu, \mathcal{P}(\mathcal{H}_b))$ by a certain Hilbert space F^b_X.*

One of our aims is a geometric realization of a Hilbert C*-module. We illustrate the two-step fibration structure of the atomic bundle as follows:



Next, we reconstruct X from $\mathcal{E}_X$. Define the space of bounded sections

$$\Gamma(\mathcal{E}_X) \equiv \{s : \mathcal{P} \to \mathcal{E}_X \,|\, \Pi_X \circ s = id_{\mathcal{P}},\ \|s\| < \infty\}$$

where the norm $\|\cdot\|$ is defined by

$$\|s\| \equiv \sup_{\rho \in \mathcal{P}} \|s(\rho)\|_\rho. \tag{1.7}$$

By standard operations, $\Gamma(\mathcal{E}_X)$ is a complex linear space. By Theorem 1.5, we can consider the differentiability of $s \in \Gamma(\mathcal{E}_X)$ at each B-fiber in the sense of Fréchet differentiability of Hilbert manifolds. Denote $\Gamma_\infty(\mathcal{E}_X)$ the set of all B-fiberwise smooth sections in $\Gamma(\mathcal{E}_X)$. Define the hermitian metric H [12] on $\Gamma_\infty(\mathcal{E}_X)$ by

$$H_\rho(s, s') \equiv \langle\, s(\rho) \,|\, s'(\rho)\,\rangle_\rho \quad (\rho \in \mathcal{P},\ s, s' \in \Gamma_\infty(\mathcal{E}_X)). \tag{1.8}$$

By these preparations, we state the following theorem which is a version of the Serre-Swan theorem generalized to non-commutative C*-algebras.

Theorem 1.6 *Let $\mathcal{A}$ be a unital C*-algebra with $(\mathcal{P}, p, B)$ in Definition 1.2, f in (1.2) and $\mathcal{K}_u(\mathcal{P})$ in (1.4). Let X be a Hilbert $\mathcal{A}$-module with $(\mathcal{E}_X, \Pi_X, \mathcal{P})$ in Definition 1.4 and H in (1.8). Then the following holds:*

(i) *Let $X \times \mathcal{P}$ be the trivial bundle over $\mathcal{P}$ and define the linear map $(P_X)_*$ from $\Gamma(X \times \mathcal{P})$ to $\Gamma(\mathcal{E}_X)$ by $\{(P_X)_*(s)\}(\rho) \equiv [s(\rho)]_\rho$ for $s \in \Gamma(X \times \mathcal{P})$, $\rho \in \mathcal{P}$. Define the subspace Γ_X of $\Gamma(\mathcal{E}_X)$ by*

$$\Gamma_X \equiv (P_X)_*(\Gamma_{const}(X \times \mathcal{P}))$$

where $\Gamma_{const}(X \times \mathcal{P})$ is the set of all constant sections of $X \times \mathcal{P}$. Then any element in Γ_X is holomorphic.

(ii) *There is a flat connection D on $\mathcal{E}_X$ such that Γ_X is a Hilbert $\mathcal{K}_u(\mathcal{P})$-module with respect to the following right $*$-action*

$$s * l \equiv s \cdot l + \sqrt{-1} D_{X_l} s \qquad (\,(s,l) \in \Gamma_X \times \mathcal{K}_u(\mathcal{P})\,) \tag{1.9}$$

and the C^*-inner product $H|_{\Gamma_X \times \Gamma_X}$.

(iii) *Under the identification $\mathcal{K}_u(\mathcal{P})$ with $\mathcal{A}$ by f, the Hilbert $\mathcal{A}$-module Γ_X is isomorphic to X.*

Here we summarize correspondences between geometry and algebra.

Gel'fand representation

	space	algebra
CG	Ω	$C(\Omega)$ pointwise product
NCG	$\mathcal{P} \to B$	$\mathcal{K}_u(\mathcal{P})$ $*$-product

Serre-Swan theorem

	vector bundle	module
CG	$E \to \Omega$	$\Gamma(E)$ pointwise action
NCG	$\mathcal{E}_X \to \mathcal{P}$	Γ_X $*$-action

where we call respectively, CG = commutative geometry as a geometry associated with commutative C*-algebras, and NCG = non-commutative geometry as a geometry associated with non-commutative C*-algebras according to [5]. In this way, NCG's are realized as visible geometries with points.

In § 2, we review the Hopf bundle and the uniform Kähler bundle. In § 2.3, we review [4] more closely. In § 3, we show Theorem 1.5. In § 4, we prove Theorem 1.6.

2 Hopf bundle and uniform Kähler bundle

2.1 The Hopf bundle and its associated bundle

We review the Hopf bundle and its associated bundle. Let $\mathbf{S} \equiv (S(\mathcal{H}), \mu, \mathcal{P}(\mathcal{H}))$ be the Hopf (fiber) bundle over a Hilbert space $\mathcal{H}$ in (1.6). The space $S(\mathcal{H})$ is

a real submanifold of $\mathcal{H}$ in the relative topology. We give $\mathcal{P}(\mathcal{H})$ the quotient topology from $\mathcal{H}\setminus\{0\}\subset\mathcal{H}$ by the natural projection. Then μ is continuous and open.

We define local trivial neighborhoods of the Hopf bundle according to Appendix C in [4]. For $h\in S(\mathcal{H})$, define

$$\begin{cases} \mathcal{V}_h\equiv\{[z]\in\mathcal{P}(\mathcal{H}):\langle h|z\rangle\neq 0\}, \quad \mathcal{H}_h\equiv\{z\in\mathcal{H}:\langle h|z\rangle=0\}, \\ \\ \beta_h:\mathcal{V}_h\to\mathcal{H}_h; \quad \beta_h([z])\equiv\langle h|z\rangle^{-1}\cdot z-h \quad ([z]\in\mathcal{V}_h). \end{cases} \tag{2.1}$$

On the holomorphic tangent space $T_\rho\mathcal{P}(\mathcal{H})$ at the local coordinate $(\mathcal{V}_h,\beta_h,\mathcal{H}_h)$ and $\beta_h(\rho)=z$, we define the Kähler metric g and the Kähler form ω on $\mathcal{P}(\mathcal{H})$ by

$$g_z^h(\bar{v},u)\equiv w_z\cdot\langle v|u\rangle-w_z^2\cdot\langle v|z\rangle\langle z|u\rangle, \quad g_z^h(u,\bar{v})\equiv g_z^h(\bar{v},u),$$
$$\omega_z^h(\bar{v},u)\equiv\sqrt{-1}\{-w_z\cdot\langle v|u\rangle+w_z^2\cdot\langle v|z\rangle\langle z|u\rangle\}, \quad \omega_z^h(u,\bar{v})\equiv-\omega_z^h(\bar{v},u)$$

for $v,u\in\mathcal{H}_h$ where $w_z\equiv 1/(1+\|z\|^2)$ and $\bar{x}\in\mathcal{H}_h^*$ means the dual vector of $x\in\mathcal{H}_h$. Then $\mathcal{P}(\mathcal{H})$ is a Kähler manifold with the holomorphic atlas $\{(\mathcal{V}_h,\beta_h,\mathcal{H}_h)\}_{h\in S(\mathcal{H})}$. For $l\in C^\infty(\mathcal{P}(\mathcal{H}))$, define the *holomorphic Hamiltonian vector field* X_l of l by the equation

$$\omega_\rho((X_l)_\rho,\overline{Y}_\rho)=\bar{\partial}_\rho l(\overline{Y}_\rho) \quad (\overline{Y}_\rho\in\overline{T}_\rho\mathcal{P}(\mathcal{H}),\,\rho\in\mathcal{P}(\mathcal{H})) \tag{2.2}$$

where $\bar{\partial}$ is the anti-holomorphic differential operator on $C^\infty(\mathcal{P}(\mathcal{H}))$ and $\overline{T}_\rho\mathcal{P}(\mathcal{H})$ denotes the anti-holomorphic tangent space of $\mathcal{P}(\mathcal{H})$ at $\rho\in\mathcal{P}(\mathcal{H})$.

The family $\{\mathcal{V}_h\}_{h\in S(\mathcal{H})}$ is a system of local trivial neighborhoods for $\mathbf{S}$ by the family $\{\psi_h\}_{h\in S(\mathcal{H})}$ of maps defined by $\psi_h:\mu^{-1}(\mathcal{V}_h)\to\mathcal{V}_h\times U(1)$;

$$\psi_h(z)\equiv(\,[z],\,\phi_h(z)\,), \quad \phi_h(z)\equiv\langle z|h\rangle\cdot|\langle h|z\rangle|^{-1}. \tag{2.3}$$

Furthermore we can verify that $\mathbf{S}$ is a principal $U(1)$-bundle.

Assume that F is a complex vector space. The fibration $\mathbf{F}\equiv(\,S(\mathcal{H})\times_{U(1)} F,\,\pi_F,\,\mathcal{P}(\mathcal{H})\,)$ is called *the associated bundle* of $\mathbf{S}$ by F if $S(\mathcal{H})\times_{U(1)}F$ is the set of all $U(1)$-orbits in the product space $S(\mathcal{H})\times F$ where the $U(1)$-action is defined by

$$(z,f)\cdot c\equiv(\bar{c}z,\,\bar{c}f) \qquad (\,c\in U(1),\,(z,f)\in S(\mathcal{H})\times F\,),$$

and the projection π_F from $S(\mathcal{H})\times_{U(1)}F$ onto $\mathcal{P}(\mathcal{H})$ is defined by $\pi_F([(x,f)])\equiv\mu(x)$ where we denote $[(x,f)]$ the element in $S(\mathcal{H})\times_{U(1)}F$ containing (x,f). The topology of $S(\mathcal{H})\times_{U(1)}F$ is induced from $S(\mathcal{H})\times F$ by the natural projection.

For $h \in S(\mathcal{H})$, the local trivialization $\psi_{F,h}$ of $\mathbf{F}$ at $\mathcal{V}_h$ is defined as the map $\psi_{F,h}$ from $\pi_F^{-1}(\mathcal{V}_h)$ to $\mathcal{V}_h \times F$ by

$$\psi_{F,h}([(z,f)]) \equiv \ (\,\mu(z),\, \phi_{F,h}([(z,f)])\,), \quad \phi_{F,h}([(z,f)]) \equiv \phi_h(z)f. \tag{2.4}$$

The definition of $\psi_{F,h}$ is independent of the choice of (z,f).

2.2 Connection

Let $\mathbf{F} = (\,S(\mathcal{H}) \times_{U(1)} F,\, \pi_F,\, \mathcal{P}(\mathcal{H})\,)$ be the associated bundle of the Hopf bundle $\mathbf{S}$ by F in § 2.1. Let $\Gamma_\infty(\mathbf{F})$ be the linear space of all smooth sections of $\mathbf{F}$. A *connection* on $\mathbf{F}$ is a $\mathbf{C}$-bilinear map D from $\mathfrak{X}(\mathcal{P}(\mathcal{H})) \times \Gamma_\infty(\mathbf{F})$ to $\Gamma_\infty(\mathbf{F})$ which is $C^\infty(\mathcal{P}(\mathcal{H}))$-linear with respect to $\mathfrak{X}(\mathcal{P}(\mathcal{H}))$ and satisfies the Leibniz law with respect to $\Gamma_\infty(\mathbf{F})$:

$$D_Y(s\cdot l) = \partial_Y l \cdot s + l\cdot D_Y s \quad (Y \in \mathfrak{X}(\mathcal{P}(\mathcal{H})),\, s\in\Gamma_\infty(\mathbf{F}),\, l \in C^\infty(\mathcal{P}(\mathcal{H}))).$$

For $Y \in \mathfrak{X}(\mathcal{P}(\mathcal{H}))$, $h \in S(\mathcal{H})$ and $\rho \in \mathcal{V}_h$, we denote Y_ρ^h the corresponding tangent vector at ρ in a local chart. Assume that a connection D on $\mathbf{F}$ is written as

$$D = \partial + A.$$

According to the notation at the local chart, we obtain families $\{A_{Y,\rho}^h : Y \in \mathfrak{X}(\mathcal{P}(\mathcal{H})),\, h \in S(\mathcal{H}),\, \rho \in \mathcal{V}_h\}$ of linear maps on F such that $\partial_Y|_\rho^h + A_{Y,\rho}^h = (\partial_Y + A_Y)_\rho^h = (\partial + A)_{Y,\rho}^h$. Then we can verify that D is a connection on $\mathbf{F}$ if and only if the following holds for each $h, h' \in S(\mathcal{H})$ with $< h|h' > \neq 0$:

$$A_{Y,\rho}^{h'} = -\frac{1}{2}\frac{\langle h|Y\rangle}{\langle h|z+h'\rangle} + A_{Y,\rho}^h \quad (\,\rho \in \mathcal{V}_{h'} \cap \mathcal{V}_h\,) \tag{2.5}$$

where Y is a holomorphic tangent vector of $\mathcal{P}(\mathcal{H})$ at ρ which is realized on $\mathcal{H}_{h'}$ and $z = \beta_{h'}(\rho)$.

A connection D on $\mathbf{F}$ is *flat* if the *curvature* R of $\mathbf{F}$ with respect to D defined by $R_{Y,Z} \equiv [D_Y,\, D_Z] - D_{[Y,Z]}$, $(Y, Z \in \mathfrak{X}(\mathcal{P}(\mathcal{H}))$, vanishes.

Proposition 2.1 *For $h \in S(\mathcal{H})$ and the chart $(\mathcal{V}_h, \beta_h, \mathcal{H}_h)$ at $\rho \in \mathcal{P}(\mathcal{H})$ in (2.1), we consider the trivializing neighborhood $\mathcal{V}_h$ for the Hopf bundle. For $Y \in \mathfrak{X}(\mathcal{P}(\mathcal{H}))$, define the operator D_Y on $\Gamma_\infty(\mathbf{F})$ by*

$$(D_Y s)(\rho) \equiv (\partial_Y s)(\rho) + (A_{Y,\rho} s)(\rho) \quad (\rho \in \mathcal{P}(\mathcal{H}))$$

where $A_{Y,\rho}$ is defined as the family $\{A_{Y,\rho}^h : h \in S(\mathcal{H}),\, \rho \in \mathcal{V}_h\}$ of linear operators on F at $(\mathcal{V}_h, \beta_h, \mathcal{H}_h)$, by

$$A_{Y,\rho}^h v \equiv -\frac{1}{2}\frac{\langle \beta_h(\rho)|Y_\rho^h\rangle}{1 + \|\beta_h(\rho)\|^2}\cdot v \quad (v \in F).$$

Then this defines a flat connection D on $\mathbf{F}$.

Proof. We can verify (2.5) for $\{A^h_{Y,\rho}\}$. Hence D is a connection. Furthermore it is straightforward to show that the curvature of D vanishes. ∎

2.3 Uniform Kähler bundle

We show a geometric characterization of the set of all pure states and the spectrum of a C*-algebra according to [4].

Definition 2.2 *A triplet (E, μ, M) is called a uniform Kähler bundle if E and M are topological spaces and μ is an open, continuous surjection from E to M such that (i) the topology of E is induced by a given uniformity, (ii) each fiber $E_m \equiv \mu^{-1}(m)$ is a Kähler manifold.*

The local triviality of uniform Kähler bundle is not assumed. In general, the topological space M is neither compact nor Hausdorff.

For uniform spaces, see Chapter 2 in [2]. Two uniform Kähler bundles (E, μ, M) and $(E^{'}, \mu^{'}, M^{'})$ are *isomorphic* if there is a pair (β, ϕ) of a uniform homeomorphism β from E to $E^{'}$ and a homeomorphism ϕ from M to $M^{'}$, such that $\mu^{'}\circ\beta = \phi\circ\mu$ and any restriction $\beta|_{\mu^{-1}(m)} : \mu^{-1}(m) \to (\mu^{'})^{-1}(\phi(m))$ is a holomorphic Kähler isometry for any $m \in M$. We call (β, ϕ) a *uniform Kähler isomorphism* from (E, μ, M) to $(E^{'}, \mu^{'}, M^{'})$.

Let $(\mathcal{H}_b, \pi_b)$ be an irreducible representation of $\mathcal{A}$ belonging to $b \in B$. Then $\rho \in \mathcal{P}_b$ corresponds $[x_\rho] \in \mathcal{P}(\mathcal{H}_b)$ where $\rho = \langle x_\rho | \pi_b(\cdot) x_\rho \rangle$. Define the bijection τ^b from $\mathcal{P}_b$ onto $\mathcal{P}(\mathcal{H}_b)$ by

$$\tau^b(\rho) \equiv [x_\rho] \quad (\rho \in \mathcal{P}_b). \tag{2.6}$$

Then $\mathcal{P}_b$ has a Kähler manifold structure induced by τ^b. Furthermore the following holds.

Theorem 2.3 (i) *For a unital C*-algebra $\mathcal{A}$, let $(\mathcal{P}, p, B)$ be as in Definition 1.2 and assume that B is endowed with the Jacobson topology [14]. Then $(\mathcal{P}, p, B)$ is a uniform Kähler bundle.*

(ii) *Let $\mathcal{A}_i$ be a C*-algebra with the associated uniform Kähler bundle $(\mathcal{P}_i, p_i, B_i)$ for $i = 1, 2$. Then $\mathcal{A}_1$ and $\mathcal{A}_2$ are * isomorphic if and only if $(\mathcal{P}_1, p_1, B_1)$ and $(\mathcal{P}_2, p_2, B_2)$ are isomorphic as uniform Kähler bundle.*

Proof. (i) See [1, 4]. (ii) See Corollary 3.3 in [4]. ∎

By Theorem 2.3 (ii), the uniform Kähler bundle $(\mathcal{P}, p, B)$ associated with $\mathcal{A}$ is uniquely determined up to uniform Kähler isomorphism.

By the above results, we obtain a fundamental correspondence between algebra and geometry as follows:

unital commutative C*-algebra	$\Leftrightarrow$	compact Hausdorff space
$\cap$		$\cap$
unital generally non-commutative C*-algebra	$\Leftrightarrow$	uniform Kähler bundle associated with a C*-algebra

The upper correspondence above is just the Gel'fand representation of unital commutative C*-algebras. By these correspondences, we show the infinitesimal version of the Takesaki duality of Hamiltonian vector fields on a symplectic manifold [10].

3 Proof of Theorem 1.5

In this section, we construct the typical fiber F_X^b of $\mathcal{E}_X$ in Theorem 1.5 and show the isomorphism among vector bundles.

In order to construct the typical fiber F_X^b of $\mathcal{E}_X$, we define the action $T = (t, \chi)$ of the group $G \equiv \mathcal{U}(\mathcal{A})$ of all unitaries in $\mathcal{A}$ on $(\mathcal{E}_X, \Pi_X, \mathcal{P})$ as follows: The action χ of G on the base space $\mathcal{P}$ is defined by

$$\chi_u(\rho) \equiv \rho \circ \mathrm{Ad}u^* \quad (u \in G,\ \rho \in \mathcal{P}).$$

The action t of G on the total space $\mathcal{E}_X$ is defined by

$$t_u([\xi]_\rho) \equiv [\,\xi u^*\,]_{\chi_u(\rho)} \quad (u \in G,\ [\xi]_\rho \in \mathcal{E}^o_{X,\rho}).$$

It is well-defined on the whole $\mathcal{E}_X$. We see that $T = (t, \chi)$ is an action of G on $(\mathcal{E}_X, \Pi_X, \mathcal{P})$ by bundle automorphism. This action also preserves B-fibers $(\mathcal{E}_X^b, \Pi_X^b, \mathcal{P}_b)$ for each $b \in B$.

For $b \in B$, let $(\mathcal{H}, \pi)$ be a representative of b. We identify $\mathcal{P}_b$ with $\mathcal{P}(\mathcal{H})$ by τ_b in (2.6). Furthermore we identify $\pi(u)$ with u for each $u \in G$. For the atomic bundle $(\mathcal{E}_X^b, \Pi_X^b, \mathcal{P}_b)$ and the Hopf bundle $(S(\mathcal{H}), \mu_b, \mathcal{P}_b)$ in (1.6), define their fiber product $\mathcal{E}_X^b \times_{\mathcal{P}_b} S(\mathcal{H})$ by

$$\mathcal{E}_X^b \times_{\mathcal{P}_b} S(\mathcal{H}) = \{(x, h) \in \mathcal{E}_X^b \times S(\mathcal{H}) : \Pi_X^b(x) = \mu_b(h)\}.$$

Thus the action σ^b of G on $\mathcal{E}_X^b \times_{\mathcal{P}_b} S(\mathcal{H})$ is defined by

$$\sigma_u^b(x,h) \equiv (t_u(x), \pi_b(u)h) \qquad ((x,h) \in \mathcal{E}_X^b \times_{\mathcal{P}_b} S(\mathcal{H}),\, u \in G).$$

Define

$$F_X^b \text{ the set of all orbits of } G \text{ in } \mathcal{E}_X^b \times_{\mathcal{P}_b} S(\mathcal{H})$$

and let $\mathcal{O}(x,h) \in F_X^b$ be the orbit of G containing $(x,h) \in \mathcal{E}_X^b \times_{\mathcal{P}_b} S(\mathcal{H})$. We see that $\mathcal{O}(0,h) = \{(0,h^{'}) : h^{'} \in S(\mathcal{H})\}$. We introduce the Hilbert space structure on F_X^b as follows: For $h \in S(\mathcal{H})$, define the sum and the scalar product on F_X^b by

$$a\mathcal{O}(x,h) + b\mathcal{O}(y,h) \equiv \mathcal{O}(ax+by,h) \quad (a,b \in \mathbf{C},\, x,y \in \mathcal{E}_X^b).$$

Then this operation is independent in the choice of x,y and h. For $h \in S(\mathcal{H})$, define the inner product $\langle\cdot|\cdot\rangle$ on the vector space F_X^b by

$$\langle \mathcal{O}(x,h)|\mathcal{O}(y,h)\rangle \equiv \langle x|y\rangle_\rho \quad (x,y \in \mathcal{E}_X^b)$$

where $\rho = \mu_b(h)$. Then $\langle \mathcal{O}(x,h)|\mathcal{O}(y,h)\rangle$ is independent in the choice of x,y,ρ and h. For $h_0 \in S(\mathcal{H})$ with $\mu_b(h_0) = \rho$, define the map R_ρ from $\mathcal{E}_{X,\rho}$ to F_X^b by $R_\rho(x) \equiv \mathcal{O}(x,h_0)$ for $x \in \mathcal{E}_{X,\rho}$. Then R_ρ is a unitary from $\mathcal{E}_{X,\rho}$ to F_X^b for each $\rho \in \mathcal{P}_b$. In this way, F_X^b is a Hilbert space.

We introduce the Hilbert bundle isomorphism in Theorem 1.5. Let $\mathbf{F}_X^b \equiv (S(\mathcal{H}) \times_{U(1)} F_X^b,\, \pi_{F_X^b},\, \mathcal{P}(\mathcal{H})\,)$ be the associated bundle of $(S(\mathcal{H}), \mu_b, \mathcal{P}(\mathcal{H}))$ by F_X^b.

Lemma 3.1 *Any element of $S(\mathcal{H}) \times_{U(1)} F_X^b$ can be written as $[(h, \mathcal{O}(x,h))]$ where $\mathcal{O}(x,h) \in F_X^b$.*

Proof. By definition of the associated bundle in § 2.1, an element of $S(\mathcal{H})\times_{U(1)} F_X^b$ is the $U(1)$-orbit $[(h, \mathcal{O}(y,k))]$. Because $(\mathcal{H}, \pi)$ is an irreducible representation of $\mathcal{A}$, the action of G on $S(\mathcal{H})$ is transitive. By this and definition of $\mathcal{O}(y,k)$, there is $u \in G$ such that $h = uk$ and $(t_u^b(y), h) \in \mathcal{O}(y,k)$. Denote $x \equiv t_u(y)$. Then $\mathcal{O}(x,h) = \mathcal{O}(y,k)$. Hence $[(h, \mathcal{O}(y,k))] = [(h, \mathcal{O}(x,h))]$. ∎

Proof of Theorem 1.5. By Lemma 3.1, we shall denote

$$[h,x] \equiv [(h, \mathcal{O}(x,h))] \in S(\mathcal{H}) \times_{U(1)} F_X^b \qquad (h \in S(\mathcal{H}),\, x \in \mathcal{E}_X^b).$$

Define the map Φ^b from $\mathcal{E}_X^b$ to $S(\mathcal{H}) \times_{U(1)} F_X^b$ by

$$\Phi^b(x) \equiv [h_x, x] \quad (x \in \mathcal{E}_X^b)$$

where $h_x \in \mu_b^{-1}(\Pi_X^b(x))$. By definition of F_X^b, the map Φ^b is bijective. We obtain a set-theoretical isomorphism (Φ^b, τ^b) of fibrations between $(\mathcal{E}_X^b, \Pi_X^b, \mathcal{P}_b)$ and $\mathbf{F}_X^b$ such that any restriction $\Phi^b|_{\mathcal{E}_{X,\rho}}$ of Φ^b at a fiber $\mathcal{E}_{X,\rho}$ is a unitary from $\mathcal{E}_{X,\rho}$ to $\pi_{F_X^b}^{-1}(\rho)$ for $\rho \in \mathcal{P}_b$. This unitary induces the Hilbert bundle isomorphism from $(\mathcal{E}_X^b, \Pi_X^b, \mathcal{P}_b)$ to $\mathbf{F}_X^b$. ∎

4 Proof of Theorem 1.6

Let us summarize our notations. Let $\mathcal{A}$ be a unital C*-algebra with the uniform Kähler bundle $(\mathcal{P}, p, B)$ and let X be a Hilbert C*-module over $\mathcal{A}$ with the atomic bundle $\mathcal{E}_X = (\mathcal{E}_X, \Pi_X, \mathcal{P})$.

Fix $b \in B$ and assume that $(\mathcal{H}, \pi)$ is a representative of b. For the Hilbert space $\mathcal{H}$, let $\{(\mathcal{V}_h, \beta_h, \mathcal{H}_h)\}_{h \in S(\mathcal{H})}$ be as in (2.1). For $\rho \in \mathcal{V}_h$, define the vector Ω_ρ^h in $\mathcal{H}$ by

$$\Omega_\rho^h \equiv \{1 + \|\beta_h(\rho)\|^2\}^{-1/2} \cdot \{\beta_h(\rho) + h\}.$$

Then $\rho = \langle \Omega_\rho^h | \pi(\cdot) \Omega_\rho^h \rangle$ and $\langle h | \Omega_\rho^h \rangle > 0$. We prepare two lemmata to prove Theorem 1.6.

Lemma 4.1 *For $s \in \Gamma(\mathcal{E}_X)$, assume that there is a family $\{\xi_\rho \in X : \rho \in \mathcal{P}\}$ such that $s(\rho) = [\xi_\rho]_\rho \in \mathcal{E}_{X,\rho}$ for each $\rho \in \mathcal{P}$ and we identify $\mathcal{E}_X^b$ with $S(\mathcal{H}) \times_{U(1)} F_X^b$ by Theorem 1.5. Let $z = \beta_h(\rho)$ for $h \in S(\mathcal{H})$ such that $\rho \in \mathcal{V}_h$. Define $w_z \equiv 1/(1 + \|z\|^2)$ and let $\phi_{F,h}$ be as in (2.4) for $F = F_X^b$. Then the following equations hold:*

$$\langle\, e \,|\, \phi_{F,h}(s(\rho)) \,\rangle = \sqrt{w_z} \cdot \langle \Omega_{\rho'}^h | \pi(\langle \xi' | \xi_\rho \rangle)(z + h) \rangle, \tag{4.1}$$

$$\partial_Y \phi_{F,h}(s(\rho)) = \mathcal{O}(\, [\partial_Y \hat{\xi}_\rho + \xi_\rho (K_{Y,\rho}^h - 2^{-1} w_z \langle z | Y \rangle)]_\rho,\, h\,) \tag{4.2}$$

for $e = \mathcal{O}([\xi']_{\rho'}, h) \in F_X^b$ where $K_{Y,\rho}^h \in \mathcal{A}$ is defined by

$$\pi(K_{Y,\rho}^h)(h + z) = Y \tag{4.3}$$

and $[\partial_Y \hat{\xi}_\rho]_\rho \in \mathcal{E}_{X,\rho}$ is defined by $\langle\, [\eta]_\rho \,|\, [\partial_Y \hat{\xi}_\rho]_\rho \,\rangle_\rho \equiv \rho(\partial_Y \langle \eta | \xi_\rho \rangle)$ for $[\eta]_\rho \in \mathcal{E}_{X,\rho}$.

Proof. By definition, we have that $\phi_{F,h}(s(\rho)) = c_{z,h} \cdot \mathcal{O}([\xi_\rho]_\rho, z)$ where $c_{z,h} \equiv \langle z | h \rangle \cdot |\langle h | z \rangle|^{-1}$. We have

$$\langle\, e \,|\, \phi_{F,h}(s(\rho)) \,\rangle = c_{z,h} \langle \mathcal{O}([\xi']_{\rho'}, h) | \mathcal{O}([\xi_\rho]_\rho, z_\rho) \rangle.$$

Let $u \in G$ such that $\pi(u^*)z = h = \Omega^h_{\rho'}$. Then $\mathcal{O}([\xi_\rho]_\rho, z) = \mathcal{O}([\xi_\rho u]_{\rho'}, \pi(u^*)z)$. By this,

$$\langle \mathcal{O}([\xi']_{\rho'}, h)|\mathcal{O}([\xi_\rho]_\rho, z_\rho)\rangle = \langle \Omega^h_{\rho'}|\pi_b(\langle \xi'|\xi_\rho\rangle)\pi_b(u)\Omega^h_{\rho'}\rangle = \langle \Omega^h_{\rho'}|\pi_b(\langle \xi'|\xi_\rho\rangle)z_\rho\rangle.$$

Because $z_\rho = c_{h,z}\Omega^h_\rho$, (4.1) is verified.

By (4.1), we get

$$\begin{aligned}\langle\, e \,|\, \partial_Y \phi_{F,h}(s(\rho)) \,\rangle = \;& \sqrt{w_z} \cdot [\langle \Omega^h_{\rho'}|\pi(\partial_Y \langle \xi'|\xi_\rho\rangle)(z+h)\rangle + \langle \Omega^h_{\rho'}|\pi(\langle \xi'|\xi_\rho\rangle)Y\rangle] \\ & -2^{-1} w_z^{3/2} \cdot \langle \Omega^h_{\rho'}|\pi(\langle \xi'|\xi_\rho\rangle)(z+h)\rangle\langle z|Y\rangle.\end{aligned}$$

Hence we obtain (4.2). ∎

For $\xi \in X$, define the section s_ξ of $\mathcal{E}_X$ by $s_\xi(\rho) \equiv [\xi]_\rho$ for $\rho \in \mathcal{P}$. Then $\|s_\xi\| = \|\xi\|$ for every $\xi \in X$. Define the linear isometry Ψ from X into $\Gamma(\mathcal{E}_X)$ by

$$\Psi(\xi) \equiv s_\xi \quad (\xi \in X).$$

Lemma 4.2 (i) *For each $\xi \in X$, $\Psi(\xi)$ belongs to $\Gamma_\infty(\mathcal{E}_X)$ and is holomorphic.*

(ii) *According to Theorem 1.5, define the connection D on $\mathcal{E}_X$ by the one in Proposition 2.1 at each fiber. Let $*$ be as in (1.9) with respect to D. Then $\Psi(\xi) * f_A = \Psi(\xi \cdot A)$ for $\xi \in X$ and $A \in \mathcal{A}$.*

Proof. Let $\rho \in \mathcal{P}_b$ for $b \in B$. Choose as a representative for b an irreducible representation $(\mathcal{H}, \pi)$. Fix $h \in S(\mathcal{H})$ and, using the notations in (2.4), take the local trivialization $\psi_{F,h}$ of the Hopf bundle at $(\mathcal{V}_h, \beta_h, \mathcal{H}_h)$ with $\rho \in \mathcal{V}_h$. Let $z \equiv \beta_h(\rho) \in \mathcal{H}_h$ and $w_z \equiv 1/(1 + \|z\|^2)$.
(i) Applying (4.2) for $s = s_\xi$, we obtain

$$\partial_Y \phi_{F,h}(s_\xi(\rho)) = \mathcal{O}([\partial_Y \hat{\xi} + \xi(K^h_{Y,\rho} - 2^{-1} w_z \cdot \langle z|Y\rangle)]_\rho,\, h\,). \tag{4.4}$$

Owing to (4.3), the right-hand side of (4.4) is smooth with respect to z. Hence s_ξ is smooth at $\mathcal{P}_b$ for each $b \in B$. For $\rho_0 \in \mathcal{P}_b$, we can choose $h_0 \in S(\mathcal{H})$ such that $\rho_0 = \langle h_0|\pi(\cdot)h_0\rangle$. Then $\beta_{h_0}(\rho_0) = 0$. According to the proof of Lemma 4.1, we have

$$\langle\, e \,|\, \phi_{F,h_0}(\rho)(s_\xi(\rho)) \,\rangle = \sqrt{w_z}\langle\, \Omega^{h_0}_{\rho'} \,|\, \pi(\langle \xi'|\xi\rangle)(z + h_0)\,\rangle$$

for $z = \beta_{h_0}(\rho)$, $\rho \in \mathcal{V}_{h_0}$. For an anti-holomorphic tangent vector $\bar{Y}$ of $\mathcal{P}_b$, we have

$$\bar{\partial}_{\bar{Y}}\phi_{F,h}\,(s_\xi(\rho)) = \mathcal{O}([-2^{-1}w_z\langle Y|z\rangle \cdot \xi]_\rho,\, h\,)$$

from which follows $\bar{\partial}_{\bar{Y}}\phi_{F,h}(\rho)\,(s_\xi(\rho))\big|_{z=0} = 0$. We see that the anti-holomorphic derivative of s_ξ vanishes at each point in $\mathcal{P}_b$. Hence s_ξ is holomorphic.
(ii) For $z \in \mathcal{H}_h$, we have

$$\{f_A \circ \beta_h^{-1}\}(z) = w_z \cdot \langle (z+h)|\pi(A)(z+h)\rangle.$$

Then the representation $X^h_{f_A}$ of the Hamiltonian vector field X_{f_A} of f_A at $(\mathcal{V}_h, \beta_h, \mathcal{H}_h)$ is

$$(X^h_{f_A})_z = -\sqrt{-1}\{\pi(A)(z+h) - \langle h|\pi(A)(z+h)\rangle(z+h)\} \quad (z \in \mathcal{H}_h).$$

If we take h such that $\beta_h(\rho_0) = 0$, then it holds that

$$(X^h_{f_A})_0 = -\sqrt{-1}\{\pi(A)h - \langle h|\pi(A)h\rangle h\}.$$

The connection D satisfies $\langle\, v\,|\,(D_{X_{f_A}}s)(\rho_0)\,\rangle_{\rho_0} = \partial_{\rho_0}(\langle v|s(\cdot)\rangle_{\rho_0})(X_{f_A})$ for $v \in \mathcal{E}_{X,\rho_0}$ and $s \in \Gamma_\infty(\mathcal{E}_X)$. Hence we have $(D_{X_{f_A}}s_\xi)(\rho_0) = [\,\xi a_{X_{f_A},0}\,]_{\rho_0}$ where $a_{X_{f_A},0} \in \mathcal{A}$ satisfies that

$$\pi(a_{X_{f_A},0})h = X_{f_A} = -\sqrt{-1}(\,\pi(A) - \langle h|\pi(A)h\rangle\,)h.$$

Therefore we have $\sqrt{-1}(D_{X_{f_A}}s_\xi)(\rho_0) = s_{\xi A}(\rho_0) - s_\xi(\rho_0)f_A(\rho_0)$ from which follows

$$(s_\xi * f_A)(\rho_0) = s_\xi(\rho_0)f_A(\rho_0) + \sqrt{-1}(D_{X_{f_A}}s_\xi)(\rho_0) = s_{\xi A}(\rho_0).$$

Therefore we obtain the statement. ∎

Finally, we come to prove Theorem 1.6.

Proof of Theorem 1.6. (i) By definition, we see that $\Gamma_X = \Psi(X)$. Therefore the statement follows from Lemma 4.2 (i).
(ii) Because $\Gamma_X = \Psi(X)$, $\mathcal{K}_u(\mathcal{P}) = f(\mathcal{A})$ and Lemma 4.2 (ii) for D, the linear space Γ_X is a right $\mathcal{K}_u(\mathcal{P})$-module.

Because $\rho(\langle\xi|\xi'\rangle) = f_{\langle\xi|\xi'\rangle}(\rho)$, we see that $H(\Psi(\xi), \Psi(\xi')) = f_{\langle\xi|\xi'\rangle} \in \mathcal{K}_u(\mathcal{P})$. Hence $H(s, s') \in \mathcal{K}_u(\mathcal{P})$ for each $s, s' \in \Gamma_X$. For $\xi, \eta \in X$ and $A \in \mathcal{A}$, we can verify that $H_\rho(\,s_\eta,\, s_\xi * f_A\,) = \{H(s_\eta, s_\xi) * f_A\}(\rho)$ where we use $H_\rho\,(\Psi(\xi), \Psi(\eta)) = \rho(\langle\xi|\eta\rangle)$ for $\xi, \eta \in X$ and $\rho \in \mathcal{P}$. Hence $H(s,\, s' * l) =$

$H(s,\, s^{'}) * l$ for each $s, s^{'} \in \Gamma_X$ and $l \in \mathcal{K}_u(\mathcal{P})$. From the property of the $\mathcal{A}$-valued inner product of X and by the proof of Lemma 4.2 (i), we obtain $\|H(s,s)\|^{1/2} = \|s\|$ for each $s \in \Gamma_X$ where the norm of $H(s,s)$ is the one defined in (1.3). Hence the statement holds.
(iii) Because $H(\Psi(\xi),\, \Psi(\xi^{'})) = f_{\langle \xi | \xi' \rangle}$, the map Ψ is an isometry from X onto Γ_X. Rewrite module actions ϕ and ψ on X and Γ_X, respectively, by

$$\phi(\xi, A) \equiv \xi A, \quad \psi(s,l) \equiv s * l \quad (\xi \in X,\, A \in \mathcal{A},\, s \in \Gamma_X,\, l \in \mathcal{K}_u(\mathcal{P})).$$

Then we obtain that $\psi \circ (\Psi \times f) = \Psi \circ \phi$ by Lemma 4.2 (ii). Hence the statement holds. ∎

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Appendix

A Example of uniform Kähler bundle

Example A.1 Assume that $\mathcal{H}$ is a separable infinite dimensional Hilbert space.

(i) Let $\mathcal{A} \equiv \mathcal{L}(\mathcal{H})$ be the C*-algebra of all bounded linear operators on $\mathcal{H}$. The uniform Kähler bundle of $\mathcal{A}$ is $(\mathcal{P}(\mathcal{H}) \cup \mathcal{P}_-,\, p,\, 2^{[0,1]} \cup \{b_0\})$ where $\mathcal{P}(\mathcal{H})$ is the projective Hilbert space of $\mathcal{H}$, $\mathcal{P}_-$ is the union of a family of projective Hilbert spaces indexed by the power set of the closed interval $[0,1]$ and $\{b_0\}$ is the one-point set corresponding to the equivalence class of identity representation $(\mathcal{H}, id_{\mathcal{L}(\mathcal{H})})$ of $\mathcal{L}(\mathcal{H})$ on $\mathcal{H}$. Since the primitive spectrum of $\mathcal{L}(\mathcal{H})$ is a two-point set, the topology of $2^{[0,1]} \cup \{b_0\}$ is equal to $\{\, \emptyset,\, 2^{[0,1]},\, \{b_0\},\, 2^{[0,1]} \cup \{b_0\}\, \}$ [8]. In this way, the base space of the uniform Kähler bundle is not always a singleton when the C*-algebra is type I.

(ii) For the C*-algebra $\mathcal{A}$ generated by the Weyl form of the 1-dimensional canonical commutation relation $U(s)V(t) = e^{\sqrt{-1}st}V(t)U(s)$ for $s, t \in \mathbf{R}$, its uniform Kähler bundle is $(\mathcal{P}(\mathcal{H}), p, \{1pt\})$. The spectrum is a one-point set $\{1pt\}$ from von Neumann uniqueness theorem [3].

(iii) The *CAR algebra* $\mathcal{A}$ is a UHF algebra with the nest $\{M_{2^n}(\mathbf{C})\}_{n \in \mathbf{N}}$. The uniform Kähler bundle has the base space $2^{\mathbf{N}}$ and each fiber on

$2^{\mathbf{N}}$ is a separable infinite dimensional projective Hilbert space where $2^{\mathbf{N}}$ is the power set of the set $\mathbf{N}$ of all natural numbers with trivial topology, that is, the topology of $2^{\mathbf{N}}$ is just $\{\emptyset, 2^{\mathbf{N}}\}$. In general, the Jacobson topology of the spectrum of a simple C*-algebra is trivial [8].

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